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Iterates of Markov Operators*

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In this paper we consider iterates of Markov operators of the form

$$\Phi f(x) = \sum_{j=0}^{m} f(j/m)\varphi_j(x)$$

where the φ_j's are linearly independent, nonnegative and sum to 1. We define the evaluation matrix of Φ to be $\Phi^* = [\varphi_j(i/m)]$ and prove that the iterates of the operator converge in the operator norm if and only if the powers of the evaluation matrix converge. Utilizing results from the theory of Markov chains we obtain explicit expressions for the limiting operator when it exists. Finally, we apply these results to Bernstein operators and then to B-spline operators.

1. INTRODUCTION

Let m be a fixed positive integer and consider an $(m+1)$-dimensional subspace $S \subset C([0, 1])$. Let $\phi_0, \ldots, \phi_m$ be a basis for S and define a linear operator $\Phi_m : C([0, 1]) \to S$ by

$$\Phi_m f(x) = \sum_{j=0}^{m} f\left(\frac{j}{m}\right) \phi_j(x). \tag{1.1}$$

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We assume

$$\sum_{j=0}^{m} \phi_j(x) \equiv 1, \qquad \phi_j(x) \geqslant 0 \tag{1.2}$$

so that Φ_m is a Markov operator.

If S is the space of polynomials of degree at most m and

$$\phi_j(x) = \binom{m}{j} x^j(1-x)^{m-j} \tag{1.3}$$

then Φ_m is the Bernstein operator of degree m so that $\Phi_m f$ is the Bernstein polynomial for f of degree (at most) m. Now, it is well known that for $f \in C([0, 1])$

$$\| \Phi_m f - f \|_\infty \to 0 \qquad (m \to \infty). \tag{1.4}$$

Kelisky and Rivlin [9] have considered iterates of the Bernstein operators and proved that for fixed m and $f \in C([0, 1])$,

$$\| \Phi_m^k f - Lf \|_\infty \to 0 \qquad (k \to \infty) \tag{1.5}$$

where

$$Lf(x) = (1-x) f(0) + x f(1). \tag{1.6}$$

Micchelli [12] has also studied the limiting behavior of the Bernstein operators by employing semigroup methods.

In this paper we consider the iterates of an arbitrary Markov operator of the form (1.1) and determine necessary and sufficient conditions for the convergence of the operator (in the operator norm). Our results permit an explicit determination of the limiting operator when it exists.

As a first application we obtain an alternate proof of (1.5). In fact, it follows readily from our results that the stronger statement $\| \Phi_m^k - L \|_\infty \to 0$ as $k \to \infty$, obtains.

Next we apply our results to obtain the limiting behavior of iterates of certain B-spline operators. B-spline operators and their resulting approximations are generalizations of the Bernstein polynomial approximations. They were first introduced by Schoenberg [15] and subsequently studied by Marsden and Schoenberg [11] and Marsden [10].

Riesenfeld [13] has incorporated B-spline approximations into the area of curve and surface design. In this context, parametrized curves consisting of B-splines are generalizations of the approximations first proposed by Bézier [1, 2] and later discussed by Forrest [7] and Gordon and Riesenfeld [8].

2. Convergence of the Iterates

The notation employed will be as in Section 1 except that the dependence upon the fixed value m will be suppressed.

Definition. If Φ is given by (1.1) then the *evaluation matrix* for Φ is the stochastic matrix

$$\Phi_* = [\phi_j(i/m)]_{i,j} . \tag{2.1}$$

For convenience we set

$$\hat{\phi} = [\phi_0 , \phi_1 ,..., \phi_m]$$

and

$$\mathbf{f} = [f(0/m), f(1/m),..., f(m/m)]^t.$$

In the following theorem we reduce the convergence problem of the iterates of Φ to the convergence problem of the powers of the evaluation matrix Φ_*.

Theorem 1. *In order that there exist a bounded linear operator* Φ^∞ on C ([0, 1]) *with*

$$\| \Phi^k - \Phi^\infty \|_\infty \to 0 \qquad (k \to \infty) \tag{2.2}$$

it is necessary and sufficient that there exist a matrix $\Phi_*{}^\infty$ *such that*

$$\Phi_*{}^k \to \Phi_*{}^\infty \qquad (k \to \infty). \tag{2.3}$$

In this case

$$\Phi^\infty f = \hat{\phi}\Phi_*{}^\infty \mathbf{f} \tag{2.4}$$

and moreover Φ^∞ *is idempotent.*

Proof. First we establish the relationship between the iterates of Φ and the powers of Φ_*; namely,

$$\Phi^k f = \hat{\phi}\Phi_*^{k-1}\mathbf{f} \qquad (k \geqslant 1). \tag{2.5}$$

For $k = 1$ this reduces to the definition given in (1.1). Proceeding inductively we obtain

$$\begin{aligned} \Phi^{k+1} f = \Phi(\Phi^k f) &= \hat{\phi}\overrightarrow{\Phi^k f} \\ &= \hat{\phi}\Phi_* \Phi_*^{k-1}\mathbf{f} = \hat{\phi}\Phi_*{}^k \mathbf{f} \end{aligned}$$

and this establishes (2.5).

Now, assume that (2.3) holds. Define the bounded linear operator Φ^∞ on $C([0, 1])$ by (2.4). Then by (2.5)

$$\begin{aligned}
\| \Phi^{k+1} - \Phi^\infty \|_\infty &= \sup_{\|f\|_\infty \leqslant 1} \|[\Phi^{k+1} - \Phi^\infty] f \|_\infty \\
&= \sup_{\|f\|_\infty \leqslant 1} \| \hat{\phi}[\Phi_*{}^k - \Phi_*{}^\infty] \mathbf{f} \|_\infty \\
&\leqslant \sup_{\|f\|_\infty \leqslant 1} \|[\Phi_*{}^k - \Phi_*{}^\infty] \mathbf{f} \|_\infty \sum_{i=0}^{m} \| \phi_i \|_\infty \\
&= \| \Phi_*{}^k - \Phi_*{}^\infty \|_\infty \sum_{i=0}^{m} \| \phi_i \|_\infty
\end{aligned}$$

and conseqeuntly (2.2) follows. This completes the proof of the sufficiency part of the theorem.

Conversely, assume that (2.2) holds. Let $j (0 \leqslant j \leqslant m)$ be fixed but arbitrary. Denote by f_j a continuous function on [0, 1] with $f_j(i/m) = \delta_{ij}$, for $0 \leqslant i \leqslant m$. Setting $\Phi_*{}^k = [\phi_{ij}^{(k)}]$ and using (2.5) we obtain the equations

$$\Phi^{k+1} f_j = \sum_{l=0}^{m} \phi_{lj}^{(k)} \phi_l .$$

Because $\Phi^{k+1} f_j \to \Phi^\infty f$, $(k \to \infty)$, it must be that $\phi_{lj}^{(k)}$ converges (say to $\phi_{lj}^{(\infty)}$) as $k \to \infty$ for $l = 0, 1, \ldots, m$. As j was arbitrary we obtain (2.3) with $\Phi_*{}^\infty = [\phi_{ij}^{(\infty)}]$.

To show that Φ^∞ is idempotent first notice that (2.2) implies $\Phi^\infty \Phi = \Phi^\infty$. Using this along with the continuity of Φ^∞ it follows that Φ^∞ idempotent. This completes the proof of Theorem 1.

COROLLARY 1. *If* $\| \Phi^k - \Phi^\infty \|_\infty \to 0$ as $k \to \infty$ *then the limiting operator* Φ^∞ *interpolates to the following points*:

$$\overrightarrow{\Phi^\infty f} = \Phi_*{}^\infty \mathbf{f} \tag{2.6}$$

where $\Phi_*{}^\infty$ *is given by* (2.4).

Now that the convergence problem of the iterates of the Markov operator has been reduced to that of the convergence of the powers of its evaluation matrix, we can utilize the theory of Markov chains (i.e., stochastic matrices). The advantage here is that there are results which yield, explicitly, the limiting matrix. In view of (2.4) this gives an explicit determination of the limiting operator.

3. Applications

As a first application of our results we consider the case where S is the space of polynomials of degree at most m and the ϕ_j's are the Bernstein basis given by (1.3). Then Φ is the Bernstein operator of degree m. That is

$$\Phi f(x) = \sum_{j=0}^{m} f(j/m)\binom{m}{j} x^j(1-x)^{m-j} \tag{3.1}$$

the evaluation matrix for Φ is

$$\Phi_* = \left[\binom{m}{j}(i/m)^j (1-(i/m))^{m-j}\right]_{i,j}. \tag{3.2}$$

In this case the stochastic matrix Φ_* has two irreducible closed sets of states; namely, $C_1 = \{0\}$ and $C_2 = \{m\}$. All other states are transient. Clearly, the period for both C_1 and C_2 equals 1 and consequently $\Phi_*{}^k$ converges.

Since,

$$i/m = \phi_m(i/m) + \sum_{j=1}^{m-1} \phi_j(i/m)(j/m)$$

it follows that (see Feller [6]) as $k \to \infty$, $\Phi_*{}^k \to \Phi_*{}^\infty$ where,

$$\Phi_*{}^\infty = \begin{bmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \frac{m-1}{m} & 0 & \cdot & \cdot & \cdot & 0 & \frac{1}{m} \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & & \cdot & \cdot & \cdot \\ \frac{1}{m} & 0 & \cdot & \cdot & \cdot & 0 & \frac{m-1}{m} \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \end{bmatrix}.$$

By Theorem 1 we can conclude that $\| \Phi^k - \Phi^\infty \| \to \infty$, as $k \to \infty$, where

$$\Phi^\infty f(x) = \hat{\phi}(x)\, \Phi_*{}^\infty \mathbf{f}.$$

According to Corollary 1, Φ^∞ interpolates to the points $\overrightarrow{\Phi^\infty f} = \Phi_*{}^\infty \mathbf{f}$. In other words, for $0 \leqslant i \leqslant m$,

$$\Phi^\infty f(i/m) = (1-(i/m))\, f(0) + (i/m)\, f(1).$$

It follows that $\Phi^\infty = L$ where L is given by (1.6). This fact includes the result (1.5) of Kelisky and Rivlin (1967) as a special case.

THEOREM 2. *Let S denote the space of polynomials of degree at most m and $\Phi : C([0, 1]) \to S$ the Bernstein operator of degree m given by* (3.1). *Then*

$$\| \Phi^k - L \|_\infty \to 0 \qquad (k \to \infty) \tag{3.3}$$

where

$$Lf(x) = (1 - x) f(0) + x f(1). \tag{3.4}$$

For our second application we consider B-splines. For $j = 2,..., m - 2$, define

$$\begin{aligned}\phi_j(x) &= \frac{4}{m} G\left[x; \frac{j-2}{m}, \frac{j-1}{m}, \frac{j}{m}, \frac{j+1}{m}, \frac{j+2}{m}\right] \\ &= \frac{m^3}{6}\left\{\left(\frac{j-2}{m} - x\right)_+^3 - 4\left(\frac{j-1}{m} - x\right)_+^3 + 6\left(\frac{j}{m} - x\right)_+^3 \right. \\ &\quad \left. - 4\left(\frac{j+1}{m} - x\right)_+^3 + \left(\frac{j+2}{m} - x\right)_+^3\right\}\end{aligned} \tag{3.5}$$

where $G(x, y) = (y - x)_+^3$ and the square bracket notation denotes the fourth divided difference of G as a function of y. These functions are the cubic B-splines first introduced by Schoenberg and Curry and Schoenberg [4] and later discussed in a compuational context by deBoor [5] and Cox [3]. They have the properties that $\phi_j \geqslant 0$, the support of ϕ_j is contained in $[(j - 2)/m, (j + 2)/m]$ and $\sum_{j=2}^{m-2} \phi_j(x) = 1$ for $x \in [3/m, (m - 3)/m]$.

Our resulting Markov operator is different than the operator considered by Schoenberg in that we augment the functions of (3.5) with the four functions

$$\begin{aligned}\phi_0(x) &= m^3[\alpha f_1(x) - f_2(x)] \\ \phi_1(x) &= m^3[-\alpha f_1(x) + 3f_3(x) + (1/6)((3/m) - x)_+^3] \\ \phi_{m-1}(x) &= m^3[-\beta f_1(1 - x) + 3f_3(1 - x) + (1/6)(((3 - m)/m) + x)_+^3] \\ \phi_m(x) &= m^3[\beta f_1(1 - x) - f_2(1 - x)]\end{aligned} \tag{3.6}$$

where $1/3 \leqslant \alpha, \beta \leqslant 1$, and for $i = 1, 2, 3$,

$$f_i(x) = (1/4)[2^{4-i}((1/m) - x)_+^3 - ((2/m) - x)_+^3]. \tag{3.7}$$

The Markov operator obtained satisfies the conditions $\Phi f(0) = f(0)$ and $\Phi f(1) = f(1)$. This property is also shared by the Bernstein operators as defined above.

In addition the parameters α and β are included so as to correspond to end conditions for these spline functions. They are used in order to affect the derivatives of the spline Φf at the end points. In fact, we have

$$\Phi f'(0) = 3\alpha m[f(1/m) - f(0)]$$
$$\Phi f'(1) = 3\beta m[f(1) - f((m-1)/m)].$$

THEOREM 3. *For* $1/3 \leqslant \alpha, \beta \leqslant 1$ *the space* Q *spanned by* $\phi_0, \phi_1, \ldots, \phi_m$ *defined in* (3.5) *and* (3.6) *is of dimension* $m+1$ *and is equivalent to the space* S *consisting of all functions* s *such that*

(i) $s \in \mathscr{P}_3$ *on* $[(i-1)/m, i/m]$, $i = 1, 2, \ldots, m$

(ii) $s \in C^2([0, 1])$

(iii) $2s'(0) + \hat{\alpha}s'(1/m) = 3\hat{\alpha}m[s(1/m) - s(0)]$
$2s'(1) + \hat{\beta}s'((m-1)/m) = 3\hat{\beta}m[s(1) - s((m-1)/m)]$

where $\hat{\alpha} = 4\alpha/(\alpha+1)$ *and* $\hat{\beta} = 4\beta/(\beta+1)$.

Proof. In view of (i) we know that on $[i/m, (i+1)/m]$,

$$\begin{aligned} s(x) = {} & g(i+1-mx)[s((i+1)/m) - s(i/m) - ms'(1/m)] \\ & - g(mx-i)[s((i+1)/m) - s(i/m) - ms'((i+1)/m)] \\ & + s(i/m)(i+1-mx) + s((i+1)/m)(mx-i) \end{aligned}$$

where $g(x) = x^3 - x^2$, Imposing the conditions of (ii) we find that for $i = 1, \ldots, m-1$,

$$s'((i-1)/m) + 4s'(i/m) + s'((i+1)/m) = 3m[s((i+1)/m) - s((i-1)/m)].$$

If we now incorporate (iii) then it follows that

$$\begin{bmatrix} 2 & \hat{\alpha} & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 4 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 4 & 1 \\ 0 & 0 & 0 & \cdots & \hat{\beta} & 2 \end{bmatrix} \begin{bmatrix} s'(0) \\ s'(1/m) \\ s'(2/m) \\ \cdot \\ \cdot \\ \cdot \\ s'((m-1)/m) \\ s'(1) \end{bmatrix} = 3m \begin{bmatrix} \hat{\alpha}[s(1/m) - s(0)] \\ s(2/m) - s(0) \\ \cdot \\ \cdot \\ \cdot \\ s(1) - s((m-2)/m) \\ \hat{\beta}[s(1) - s((m-1)/m)] \end{bmatrix}. \tag{3.8}$$

Now, the coefficient matrix of (3.8) is diagonally dominant for $1/3 \leqslant \alpha, \beta \leqslant 1$. Moreover, for $1/3 \leqslant \alpha, \beta < 1$ it is strictly diagonally dominant. In the case where $\alpha = 1$ or $\beta = 1$ it is not strictly diagonally dominant, although at most two elementary column operations will make it so. Consequently, we deduce that in all cases the $s'(i/m)$ are uniquely determined by the $s(i/m)$, $i = 0, 1, \ldots, m$ and hence so is s.

Next, using the facts that $\phi_0(0) = 1$, $\phi_0(1/m) = (1-\alpha)/4$, $\phi_0'(0) = -3\alpha m$, $\phi_0'(1/m) = 3m(\alpha - 1)/4$, $\phi_1(0) = 0$, $\phi_1(1/m) = (7 + 3\alpha)/12$, $\phi_1'(0) = 3\alpha m$, $\phi_1'(1/m) = m(1 - 3\alpha)/4$ and analogous properties for ϕ_{m-1} and ϕ_m it can be verified that ϕ_i, $i = 0, \ldots, m$ satisfy (iii). Hence it is clear that $Q \subset S$.

In order to show that each $s \in S$ can be written as $s = \sum \alpha_i \phi_i$ it is only necessary to observe that the evaluation matrix for the ϕ_i's:

$$\Phi_* = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{3-3\alpha}{12} & \frac{3\alpha+7}{12} & \frac{1}{6} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & \frac{4}{6} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{6} & \frac{3\beta+7}{12} & \frac{3-3\beta}{12} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

is strictly diagonally dominant. This completes the proof of Theorem 3.

Now, by Theorem 1, in order to determine the limiting behavior of the iterates of Φ it is only necessary to analyze the asymptotic behavior of the powers of the evaluation matrix Φ_* given in (3.9). In order to accomplish this we consider two separate cases.

Case 1. $\alpha = \beta = 1$. In this case the stochastic matrix Φ_* consists of three irreducible closed classes; namely, $C_1 = \{0\}$, $C_2 = \{1, 2, \ldots, m-1\}$, $C_3 = \{m\}$. The class C_2 has a matrix which is doubly stochastic and it now follows that as $k \to \infty$, $\Phi_*^k \to \Phi_*^\infty$ where

$$\Phi_*^\infty = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{m-1} & \frac{1}{m-1} & \cdots & \frac{1}{m-1} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \frac{1}{m-1} & \frac{1}{m-1} & \cdots & \frac{1}{m-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Using Theorem 1 we know that $\| \Phi^k - \Phi^\infty \|_\infty \to 0$, as $k \to \infty$, where

$$\Phi^\infty f(x) = \hat{\phi}(x)\, \Phi_*^\infty \mathbf{f}.$$

By Corollary 1, Φ^∞ interpolates to the points $\Phi_*^\infty \mathbf{f}$ and hence

$$\Phi^\infty f(i/m) = (1/(m-1)) \sum_{j=1}^{m-1} f(j/m); \qquad 1 \leqslant i \leqslant m-1$$

$$\Phi^\infty f(0) = f(0) \qquad \text{and} \qquad \Phi^\infty f(1) = f(1).$$

Thus, in contrast to the Bernstein case, the limiting operator does not, in general, take f to the line segment joining $(0, f(0))$ and $(1, f(1))$.

THEOREM 4. *Let S denote the vector space of B-splines with basis given by* (3.5)-(3.7). *Assume that* $\alpha = \beta = 1$. *Define*

$$Kf(x) = [f(0) - Af]\, \phi_0(x) + [f(1) - Af]\, \phi_m(x) + Af \tag{3.10}$$

where

$$Af = (1/(m-1)) \sum_{j=1}^{m-1} f(j/m). \tag{3.11}$$

Then,

$$\| \Phi^k - K \|_\infty \to 0 \qquad (k \to \infty). \tag{3.12}$$

Case 2. $\alpha \neq 1$ *or* $\beta \neq 1$. In this case Φ_* has two irreducible closed classes; namely, $C_1 = \{0\}$ and $C_2 = \{m\}$. Results on tridiagonal stochastic matrices along with some computations show that, as $k \to \infty$,

$$\Phi_*^k \to \begin{bmatrix} 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ a_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1-a_1 \\ \cdot & \cdot & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot \\ a_{m-1} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1-a_{m-1} \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 \end{bmatrix}$$

where

$$a_i = \frac{(1-\alpha)[3(1-\beta)(m-1-i)+2]}{2(1-\beta)+3(1-\alpha)(1-\beta)(m-2)+2(1-\alpha)} \tag{3.13}$$

for $1 \leqslant i \leqslant m-1$.

Consequently, by Theorem 1, Φ^k converges and moreover by Corollary 1,

$$\Phi^\infty f(i/m) = a_i f(0) + (1 - a_i) f(1); \qquad 1 \leqslant i \leqslant m - 1$$

$$\Phi^\infty f(0) = f(0) \qquad \text{and} \qquad \Phi^\infty f(1) = f(1).$$

THEOREM 5. *Let S denote the vector space of B-splines with basis given by* (3.5)–(3.7). *Assume that either* $\alpha \neq 1$ *or* $\beta \neq 1$. *Define*

$$K_{\alpha,\beta} f(x) = f(0)\, \phi_0(x) + f(1)\, \phi_1(x) + \sum_{i=1}^{m-1} [a_i f(0) + (1 - a_i) f(1)]\, \phi_i(x) \tag{3.14}$$

where a_i *is given by* (3.13). *Then,*

$$\| \Phi^k - K_{\alpha,\beta} \|_\infty \to 0 \qquad (k \to \infty). \tag{3.15}$$

A necessary and sufficient condition for $K_{\alpha,\beta} = L$ is that $\alpha = \beta = 1/3$.

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